

WEIGHTED ESTIMATES FOR BILINEAR FRACTIONAL INTEGRAL OPERATORS AND THEIR COMMUTATORS

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ABSTRACT. In this paper we will prove several weighted estimates for bilinear fractional integral operators and their commutators with BMO functions. We also prove maximal function control theorem for these operators, that is, we prove the weighted L^p norm is bounded by the weighted L^p norm of a natural maximal operator when the weight belongs to A_∞ . As a corollary we are able to obtain new weighted estimates for the bilinear maximal function associated to the bilinear Hilbert transform.

1. INTRODUCTION

We are interested in the family of bilinear fractional integrals

$$\mathbf{Bl}_\alpha(f, g)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

The study of \mathbf{Bl}_α was initiated by Kenig and Stein in [9] and Grafakos in [6] who proved that $\mathbf{Bl}_\alpha : L^{p_1} \times L^{p_2} \rightarrow L^q$ when $1 < p_1, p_2 < \infty$ and q satisfies $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}$. The main interest of these operators is the singular nature of the kernel. In fact, \mathbf{Bl}_α has the same relationship to the bilinear Hilbert transform,

$$\mathbf{BH}(f, g)(x) = p.v. \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{y} dy$$

as the linear fractional integral has to the Hilbert transform. We aim to study weighted norm inequalities of the form

$$\mathbf{Bl}_\alpha : L^{p_1}(v_1) \times L^{p_2}(v_2) \longrightarrow L^q(u).$$

Weighted inequalities for these singular operators were unknown until the second author made some progress in [14] for the case when $p \leq q \leq 1$. The main results of [14] are stated in Theorem 2.1 in the next section.

This paper was originally an attempt to expand the range for p and q , but then as the theory developed, we were also interested in considering the effects of several types of commutators on \mathbf{Bl}_α . Given a linear operator T and a function b , the commutator $[b, T]$ is defined to be

$$[b, T]f = bT(f) - T(bf).$$

Coifman, Rochberg and Weiss [2] introduced commutators of singular integral operators as a tool to extend the classical factorization theory of H^p spaces. They

Key words and phrases. Bilinear operators, fractional operators, commutators, weighted inequalities, bump conditions.

The second author was supported by NSF Grant #1201504.

proved that if $b \in BMO$ and T is a singular integral operator, then $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Weighted estimates for the linear fractional integral operator were done by Muckenhoupt and Wheeden [15] in the one weight case. Pérez [16] proved sufficient two weight bump conditions for the boundedness of I_α . The commutator of I_α was first considered by Chanillo [1], who showed that if $b \in BMO$, then $[b, I_\alpha]$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ with $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Weighted estimates for $[b, I_\alpha]$ were studied by D. Cruz-Uribe and the second author in [3] where it was shown that if $b \in BMO$, $1 < p \leq q < \infty$ and (u, v) is a pair of weights satisfying

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{A, Q} \|v^{-\frac{1}{p}}\|_{B, Q} < \infty$$

where $A(t) = t^q \log(e + t)^{2q-1+\delta}$ and $B(t) = t^{p'} \log(e + t)^{2p'-1+\delta}$, we have

$$\|[b, I_\alpha]\|_{L^q(u)} \lesssim \|b\|_{BMO} \|f\|_{L^p(v)}.$$

When considering a bilinear operator \mathbf{BT} , we define the commutators on the first and the second components to be

$$[b, \mathbf{BT}]_1(f, g) = b \mathbf{BT}(f, g) - \mathbf{BT}(bf, g)$$

and

$$[b, \mathbf{BT}]_2(f, g) = b \mathbf{BT}(f, g) - \mathbf{BT}(f, bg).$$

Let $\vec{b} = (b_1, \dots, b_N)$, where b_i 's are given functions, and $\vec{\beta} = (\beta_1, \dots, \beta_N) \in \{1, 2\}^N$, the iterated product commutators of a bilinear operator \mathbf{BT} is defined (from inner to outer) to be

$$[\vec{b}, \mathbf{BT}]_{\vec{\beta}} = [b_N, [b_{N-1}, \dots, [b_2, [b_1, \mathbf{BT}]_{\beta_1}]_{\beta_2} \dots]_{\beta_{N-1}}]_{\beta_N}.$$

In the linear case, these type of commutators were studied by Pérez and Rivera-Rios [18]. Given a bilinear operator \mathbf{BT} , we may rearrange the commutators in any order as the following Proposition states.

Proposition 1.1. *For any permutation σ on $\{1, \dots, N\}$,*

$$(1.1) \quad [\sigma(\vec{b}), \mathbf{BT}]_{\sigma(\vec{\beta})} = [\vec{b}, \mathbf{BT}]_{\vec{\beta}}$$

where $\sigma(\vec{b}) = (b_{\sigma(1)}, \dots, b_{\sigma(N)})$ and $\sigma(\vec{\beta}) = (\beta_{\sigma(1)}, \dots, \beta_{\sigma(N)})$. In particular, equality (1.1) is valid for any permutation σ_0 be such that $\sigma_0(\vec{\beta}) = (1, \dots, 1, 2, \dots, 2)$.

For simplicity in the notation and proof, from now on we will always assume that $\vec{\beta} = (1, \dots, 1, 2, \dots, 2)$, and reserve the notation $m = m(\vec{\beta})$ to denote the number of 1's in $\vec{\beta}$.

2. MAIN RESULTS

Throughout this paper we will work with 2 different cases. The first case is when p_1 and p_2 get close enough to 1, which will force $p = \frac{p_1 p_2}{p_1 + p_2} \leq 1$, while the second case is when $p > 1$. Our departure is the following result of the second author [14].

Theorem 2.1 ([14]). *Suppose $1 < p_1, p_2$ and $q \leq 1$ are such that $\frac{1}{2} < p = \frac{p_1 p_2}{p_1 + p_2} \leq q \leq 1$. If (u, v_1, v_2) are weights satisfying*

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q u^{\frac{1}{1-q}} \right)^{\frac{1-q}{q}} \|v_1^{-\frac{1}{p_1}}\|_{\phi_1, Q} \|v_2^{-\frac{1}{p_2}}\|_{\phi_2, Q} < \infty$$

where $\phi_i(t) = t^{p'_i} \log(e+t)^{p'_i-1+\delta}$, $i \in \{1, 2\}$, and $\left(f_Q u^{\frac{1}{1-q}}\right)^{1-q} = \sup_Q u$ when $q = 1$. Then, the inequality

$$\|\mathbf{Bl}_\alpha(f, g)\|_{L^q(u)} \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

By using a different technique, we are able to prove a similar result in the case when $1 < p \leq q < \infty$.

Theorem 2.2. Suppose $0 < \alpha < n$, $p_1 > r > 1$, $p_2 > s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p \leq q < \infty$, and the set of weights (u, v_1, v_2) satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\psi, Q} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q}^{\frac{1}{r}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q}^{\frac{1}{s}} < \infty$$

where ψ, ϕ_1, ϕ_2 are Young functions satisfying $\bar{\psi} \in B_{q'}$, $\bar{\phi}_1 \in B_{\frac{p_1}{r}}$ and $\bar{\phi}_2 \in B_{\frac{p_2}{s}}$. Then the inequality

$$\|\mathbf{Bl}_\alpha(f, g)\|_{L^q(u)} \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

For the general commutators defined on \mathbf{Bl}_α , we have Theorems 2.3 and 2.4 as stated below.

Theorem 2.3. Suppose $0 < \alpha < n$, $\vec{b} \in BMO^N$, $p_1 > 1$, $p_2 > 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{2} < p \leq q \leq 1$, and the set of weights (u, v_1, v_2) satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{1-q}}\|_{\psi, Q}^{\frac{1-q}{q}} \|v_1^{-\frac{1}{p_1}}\|_{\phi_1, Q} \|v_2^{-\frac{1}{p_2}}\|_{\phi_2, Q} < \infty$$

where $\phi_1(t) = t^{p'_1} \log(e+t)^{(m+1)p'_1-1+\delta}$, $\phi_2(t) = t^{p'_2} \log(e+t)^{(N-m+1)p'_2-1+\delta}$, $\delta > 0$, and $\psi(t) = t \log(e+t)^{\frac{qN}{1-q}}$ with $\|u^{\frac{1}{1-q}}\|_{\psi, Q}^{1-q} = \sup_Q u$ when $q = 1$. Then, the inequality

$$\|[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}(f, g)\|_{L^q(u)} \lesssim \|\vec{b}\| \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$, where $\|\vec{b}\| = \prod_{i=1}^N \|b_i\|_{BMO}$.

Theorem 2.4. Suppose $0 < \alpha < n$, $\vec{b} \in BMO^N$, $p_1 > r > 1$, $p_2 > s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p \leq q < \infty$, and the set of weights (u, v_1, v_2) satisfies

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\psi, Q} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q}^{\frac{1}{r}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q}^{\frac{1}{s}} < \infty$$

where

$$\phi_1(t) = t^{\left(\frac{p_1}{r}\right)'} \log(e+t)^{(mr+1)\left(\frac{p_1}{r}\right)'-1+\delta}$$

$$\phi_2(t) = t^{\left(\frac{p_2}{s}\right)'} \log(e+t)^{((N-m)s+1)\left(\frac{p_2}{s}\right)'-1+\delta}$$

and $\psi(t) = t^q \log(e+t)^{(N+1)q-1+\delta}$, $\delta > 0$. Then, the inequality

$$\|[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}(f, g)\|_{L^q(u)} \lesssim \|\vec{b}\| \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

We note here that the Hölder pairs (r, s) in Theorems 2.2 and 2.4 exist and there are many such pairs. To name a few, we can start with $r = \frac{p_1}{p}$ and $s = \frac{p_2}{p}$, then we use the facts that $r < p_1$ and $s < p_2$ to obtain more choices by considering either $r = \frac{p_1}{p} + \epsilon$ or $s = \frac{p_2}{p} + \epsilon$, for small $\epsilon > 0$. Also, we have completed just two thirds of the whole picture (i.e. the two cases: $p \leq q \leq 1$ and $1 < p \leq q$). The case when $p \leq 1 \leq q$ is still open.

While studying \mathbf{Bl}_α and $[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}$, we need the following maximal operators: given two Young functions Φ and Ψ

$$\mathcal{M}_\alpha^{\Phi, \Psi}(f, g)(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \|f\|_{\Phi, Q} \|g\|_{\Psi, Q}.$$

When $\alpha = 0$ we write $\mathcal{M}_0^{\Phi, \Psi} = \mathcal{M}^{\Phi, \Psi}$. When $\Phi(t) = t^r$ and $\Psi(t) = t^s$ we write

$$\mathcal{M}_\alpha^{\Phi, \Psi}(f, g)(x) = \mathcal{M}_\alpha^{r, s}(f, g)(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \left(\int_Q |f|^r \right)^{\frac{1}{r}} \left(\int_Q |g|^s \right)^{\frac{1}{s}}$$

and when $\Phi(t) = \Psi(t) = t$ we write

$$\mathcal{M}_\alpha(f, g)(x) = \mathcal{M}_\alpha^{1, 1}(f, g)(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \left(\int_Q |f| \right) \left(\int_Q |g| \right).$$

The controls that we have mentioned above are stated in the two theorems below.

Theorem 2.5. *Suppose $0 < \alpha < n$, $0 < q < \infty$ and (r, s) is a Hölder pair. If the weight $w \in A_\infty$, then*

$$\int_{\mathbb{R}^n} |\mathbf{Bl}_\alpha(f, g)(x)|^q w(x) dx \lesssim \int_{\mathbb{R}^n} \mathcal{M}_\alpha^{r, s}(f, g)(x)^q w(x) dx.$$

Theorem 2.6. *Suppose $0 < \alpha < n$, $0 < q < \infty$ and (r, s) is a Hölder pair. If the weight $w \in A_\infty$, then*

$$\int_{\mathbb{R}^n} |[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)|^q w(x) dx \lesssim \|\vec{b}\|^q \int_{\mathbb{R}^n} \mathcal{M}_\alpha^{\Phi, \Psi}(f, g)(x)^q w(x) dx$$

where $\Phi(t) = t^r \log(e + t)^{mr}$ and $\Psi(t) = t^s \log(e + t)^{(N-m)s}$.

In Theorem 2.2, if we consider special power-bump Young functions, then the condition on the weights (u, v_1, v_2) become

$$(2.1) \quad \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q u \right)^{\frac{1}{q}} \left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{rp_1}} \left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{sp_2}} < \infty$$

where $\left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{r}} = (\inf_Q v_1)^{-1}$ when $p_1 = r$, and $\left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{s}} = (\inf_Q v_2)^{-1}$ when $p_2 = s$.

It turns out that condition (2.1) can be characterized via the weak type and the strong type weighted boundedness of the maximal operator $\mathcal{M}_\alpha^{r, s}$, not only for $0 < \alpha < n$ but also for $\alpha = 0$. These results are stated in the following theorems.

Theorem 2.7. *Suppose $0 \leq \alpha < n$, $p_1 \geq r > 1$, $p_2 \geq s > 1$, $1 < p = \frac{p_1 p_2}{p_1 + p_2} \leq q$. Then (u, v_1, v_2) satisfies condition (2.1) if and only if the inequality*

$$\sup_{\lambda > 0} \lambda u(\{x : \mathcal{M}_\alpha^{r, s}(f, g)(x) > \lambda\})^{\frac{1}{q}} \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

Theorem 2.8. Suppose $0 \leq \alpha < n$, $p_1 > r > 1$, $p_2 > s > 1$, $1 < p = \frac{p_1 p_2}{p_1 + p_2} \leq q$. If (u, v_1, v_2) satisfies

$$(2.2) \quad \sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\psi, Q} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q}^{\frac{1}{r}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q}^{\frac{1}{s}} < \infty$$

where ψ, ϕ_1, ϕ_2 are Young functions satisfying $\bar{\psi} \in B_{q'}$, $\bar{\phi}_1 \in B_{\frac{p_1}{r}}$ and $\bar{\phi}_2 \in B_{\frac{p_2}{s}}$, then the inequality

$$\|\mathcal{M}_\alpha^{r,s}(f, g)\|_{L^q(u)} \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

So far, we have seen that the weak type weighted boundedness for $\mathcal{M}_\alpha^{r,s}$ is equivalent to condition (2.1). Since strong type boundedness implies weak type one, it obviously implies condition (2.1). In order to get the other way around, besides the stricter requirements that $p_1 > r$ and $p_2 > s$, we have to “bump” up our condition on the weights by using the Orlicz norms as appeared in condition (2.2) of Theorem 2.8. However, things become much nicer in 1-weight settings [i.e. when $u^{\frac{1}{q}} = v_1^{\frac{1}{p_1}} v_2^{\frac{1}{p_2}}$].

Theorem 2.9. Suppose $0 \leq \alpha < n$, $p_1 > r > 1$, $p_2 > s > 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then the inequality

$$\|\mathcal{M}_\alpha^{r,s}(f, g)\|_{L^q(w_1^{\frac{q}{p_1}} w_2^{\frac{q}{p_2}})} \lesssim \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)}$$

holds if and only if the weights (w_1, w_2) satisfy

$$(2.3) \quad \sup_Q \left(\int_Q w_1^{\frac{q}{p_1}} w_2^{\frac{q}{p_2}} \right)^{\frac{1}{q}} \left(\int_Q w_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{rp_1}} \left(\int_Q w_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{sp_2}} < \infty.$$

Roughly speaking, when $p_1 > r$ and $p_2 > s$, in the multiple weight setting, (u, v_1, v_2) we have

$$\text{Strong bound for } \mathcal{M}_\alpha^{r,s} \Rightarrow \text{Weak bound for } \mathcal{M}_\alpha^{r,s} \Leftrightarrow \text{Condition (2.1)}$$

and in vector weight setting (w_1, w_2) and $u = w_1^{\frac{q}{p_1}} w_2^{\frac{q}{p_2}}$ we have

$$\text{Strong bound for } \mathcal{M}_\alpha^{r,s} \Leftrightarrow \text{Weak bound for } \mathcal{M}_\alpha^{r,s} \Leftrightarrow \text{Condition (2.3)}.$$

As an immediate consequence of Theorems 2.5 and 2.9, we have the following result.

Corollary 2.10. Under the same assumptions as in Theorem 2.9, condition (2.3) implies

$$\|\text{Bl}_\alpha(f, g)\|_{L^q(w_1^{\frac{q}{p_1}} w_2^{\frac{q}{p_2}})} \lesssim \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)}$$

for all $f \in L^{p_1}(w_1)$ and $g \in L^{p_2}(w_2)$.

Finally, we end with an application of our estimates. The associated maximal operator to the bilinear Hilbert transform is defined as

$$\text{BM}(f, g)(x) = \sup_{r>0} \frac{1}{(2r)^n} \int_{[-r, r]^n} |f(x-y)g(x+y)| dy.$$

In the one dimensional case, this operator is studied in [11], where it is shown that it satisfies

$$\text{BM} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \rightarrow L^p(\mathbb{R})$$

when $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $p > 2/3$. Surprisingly, and contrary to the usual paradigm in harmonic analysis, the boundedness of the bilinear Hilbert transform was shown first and used to prove the boundedness of BM. Other than trivial conditions on the weights (i.e., assuming separate conditions on the weights such as *both* w_1 and w_2 belong to A_p there are no known weighted estimates for BM. By Hölder's inequality we have that

$$\text{BM}(f, g)(x) \leq \mathcal{M}^{r,s}(f, g)(x)$$

for any Hölder's pair of exponents r and s and therefore have the following corollaries.

Corollary 2.11. Suppose $p_1 > r > 1$, $p_2 > s > 1$, $1 < p = \frac{p_1 p_2}{p_1 + p_2}$. If (u, v_1, v_2) satisfies

$$(2.4) \quad \sup_Q \|u^{\frac{1}{p}}\|_{\psi, Q} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q}^{\frac{1}{r}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q}^{\frac{1}{s}} < \infty$$

where ψ, ϕ_1, ϕ_2 are Young functions satisfying $\bar{\psi} \in B_{p'}$, $\bar{\phi}_1 \in B_{\frac{p_1}{r}}$ and $\bar{\phi}_2 \in B_{\frac{p_2}{s}}$, then the inequality

$$\|\text{BM}(f, g)\|_{L^p(u)} \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}$$

holds for all $f \in L^{p_1}(v_1)$ and $g \in L^{p_2}(v_2)$.

Finally we end with a one vector weight theorem. In this case we will take the natural definition of $r = \frac{p_1}{p}$ and $s = \frac{p_2}{p}$.

Corollary 2.12. Suppose $p_1, p_2 > 1$, and (w_1, w_2) are weights satisfying

$$(2.5) \quad \sup_Q \left(\int_Q w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}} \right)^{\frac{1}{p}} \left(\int_Q w_1^{\frac{1}{1-p}} \right)^{\frac{p-1}{p_1}} \left(\int_Q w_2^{\frac{1}{1-p}} \right)^{\frac{p-1}{p_2}} < \infty$$

where $\left(\int_Q w_i^{\frac{1}{1-p}} \right)^{p-1} = (\inf_Q w_i)^{-1}$ when $p = 1$, $i \in \{1, 2\}$.

Then, BM is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^{p, \infty}(w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}})$ whenever $p \geq 1$. Moreover, BM is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(w_1^{\frac{p}{p_1}} w_2^{\frac{p}{p_2}})$ whenever $p > 1$.

In this paper, we will first give proof for Theorems 2.3, 2.4 and 2.6. Theorems 2.2 and 2.5 can be proved using similar techniques as those in the proof of theorems 2.4 and 2.6, respectively. The later Theorems are in fact easier than the former and follow from the exact techniques, so we choose to leave them for interested readers. We then will sketch the proof for Theorems 2.7, 2.8 and 2.9, and end with an application of our results: a bilinear Stein-Weiss inequality.

3. PRELIMINARIES

A dyadic grid \mathcal{D} is a countable collection of cubes that satisfies the following properties:

- (1) $Q \in \mathcal{D} \Rightarrow \ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.
- (2) For each $k \in \mathbb{Z}$, the set $\{Q \in \mathcal{D} : \ell(Q) = 2^k\}$ forms a partition of \mathbb{R}^n .
- (3) $Q, P \in \mathcal{D} \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$.

One very clear example for this concept is the dyadic grid that is formed by translating and then dilating the unit cube $[0, 1]^n$ all over \mathbb{R}^n . More precisely, it is formulated as

$$\mathcal{D} = \{2^{-k}([0, 1]^n + m) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

In practice, we also make extensive use of the following family of dyadic grids.

$$\mathcal{D}^t = \{2^{-k}([0, 1]^n + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, 1/3\}^n.$$

Lerner [12] proved the following result.

Theorem 3.1. *Given any cube Q in \mathbb{R}^n , there exists a $t \in \{0, 1/3\}^n$ and a cube $Q_t \in \mathcal{D}^t$ such that $Q \subset Q_t$ and $\ell(Q_t) \leq 6\ell(Q)$.*

Next, we are going to give necessary details of Orlicz spaces. For more details, we refer the reader to [4]. A Young function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous, convex and strictly increasing function with $\Phi(0) = 0$ and $\frac{\Phi(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$. Given a Young function, there exists another Young function, denoted as $\bar{\Phi}$ and referred to as the associate function, that satisfies $t \leq \Phi^{-1}(t) \bar{\Phi}^{-1}(t) \leq 2t$ when $t > 0$. For instance, the Young function $\Phi(t) = t^p$, $p > 1$, has its associate Young function $\bar{\Phi}(t) = t^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. There are many more types of Young functions, but the most commonly seen are the “log-bump” functions $\Phi(t) = t^r \log(e + t)^s$ for some $r > 1$ and $s \in \mathbb{R}$.

The Orlicz average of f over a cube Q is given by

$$\|f\|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}$$

which is equivalent to

$$\|f\|'_{\Phi, Q} = \inf_{\lambda > 0} \left\{ \lambda + \frac{\lambda}{|Q|} \int_Q \Phi \left(\frac{|f(x)|}{\lambda} \right) dx \right\}.$$

This result is due to Krasnosel'skii and Rutickii [10]. In fact,

$$\|f\|_{\Phi, Q} \leq \|f\|'_{\Phi, Q} \leq 2\|f\|_{\Phi, Q}.$$

The Orlicz maximal function is then defined to be

$$M_{\Phi}(f)(x) = \sup_{Q \ni x} \|f\|_{\Phi, Q}.$$

Pérez [17] gave a necessary and sufficient condition for the boundedness of these Orlicz maximal operators.

Theorem 3.2. *For any $p \in (1, \infty)$,*

$$\|M_{\Phi} f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

if and only if Φ satisfies the B_p integrability condition, i.e. there exists $c > 0$ such that

$$\int_c^\infty \frac{\Phi(t)}{t^{p+1}} dt < \infty.$$

There is also a generalized Hölder inequality for these Orlicz averages.

Lemma 3.3. *If Φ, Ψ, Θ are Young functions such that*

$$\Phi^{-1}(t)\Psi^{-1}(t) \lesssim \Theta^{-1}(t), \quad \forall t \geq t_0 \geq 0$$

then

$$\|fg\|_{\Theta, Q} \lesssim \|f\|_{\Phi, Q} \|g\|_{\Psi, Q}.$$

In particular, for any Young function ψ ,

$$\int_Q |f(x)g(x)| dx \leq 2 \|f\|_{\psi, Q} \|g\|_{\bar{\psi}, Q}.$$

When $p > 1$, a weight $w \in A_p$ if and only if

$$\sup_Q \left(\int_Q w \right) \left(\int_Q w^{1-p'} \right)^{p-1} < \infty.$$

When $p = 1$, we have $w \in A_1$ if and only if

$$Mw(x) \leq Cw(x), \quad a.e. x \in \mathbb{R}^n$$

where M is the Hardy-Littlewood maximal function. Finally we define A_∞ as the union of all A_p classes for $p > 1$. Also from [5] we know the following facts.

Lemma 3.4. *If $w \in A_\infty$ then the following hold:*

- i) for every $\eta \in (0, 1)$, there exists $\kappa \in (0, 1)$ such that: given a cube Q and $S \subseteq Q$ with $|S| \leq \eta|Q|$, we will also have $w(S) \leq \kappa w(Q)$;*
- ii) there exist an $m > 1$ such that*

$$\left(\int_Q w^m \right)^{\frac{1}{m}} \leq C \int_Q w.$$

Next, we would like to briefly discuss the bilinear Muckenhoupt condition, $A_{\vec{P}, q}$ condition, which was introduced by the second author in [13]. A set of weights (w_1, \dots, w_m) is said to be in the class $A_{\vec{P}, q}$ if

$$\sup_Q \left(\int_Q (w_1 w_2)^q \right)^{\frac{1}{q}} \left(\int_Q w_1^{-p'_1} \right)^{\frac{1}{p'_1}} \left(\int_Q w_2^{-p'_2} \right)^{\frac{1}{p'_2}} < \infty.$$

The second author also proved that if $p_i \leq q_i$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then

$$\bigcup_{q_1, q_2} (A_{p_1, q_1} \times A_{p_2, q_2}) \subsetneq A_{\vec{P}, q}$$

where the inclusion was shown to be strict.

Theorem 3.5. *Suppose $1 < p_1, p_2 < \infty$, and $(w_1, w_2) \in A_{\vec{P}, q}$, then we have*

$$(w_1 w_2)^q \in A_{2q} \quad \text{and} \quad w_i^{-p'_i} \in A_{2p'_i}.$$

Cruz-Uribe, Martell and Pérez [4] proved an extrapolation theorem for A_∞ weights. Namely,

Theorem 3.6. *Suppose there exist $p_0 \in (0, \infty)$ such that*

$$\int_{\mathbb{R}^n} |f|^{p_0} w \leq C \int_{\mathbb{R}^n} |g|^{p_0} w \quad \forall w \in A_\infty$$

then we have

$$\int_{\mathbb{R}^n} |f|^p w \leq C \int_{\mathbb{R}^n} |g|^p w \quad \forall w \in A_\infty, \forall p \in (0, \infty).$$

Finally, we will need the concept of bounded mean oscillation. Let BMO denote the space of functions of bounded mean oscillation, i.e., functions b such that

$$\|b\|_{BMO} = \sup_Q \int_Q |b(x) - b_Q| dx < \infty$$

where $b_Q = \int_Q b(x) dx$.

BMO functions satisfy the exponential integrability which is a consequence of the John-Nirenberg theorem.

Theorem 3.7. *Given $b \in BMO$, there exists a constant c_n such that for every cube Q ,*

$$\sup_Q \int_Q \exp\left(\frac{|b(x) - b_Q|}{2^{n+2}\|b\|_{BMO}}\right) dx \leq c_n.$$

In particular,

$$\|b - b_Q\|_{\exp L, Q} \leq c_n 2^{n+2} \|b\|_{BMO}.$$

A proof of Theorem 3.7 can be found in [8].

Corollary 3.8. If $b \in BMO$, then for any $\xi > 0$,

$$\| |b - b_Q|^\xi \|_{\exp(L^{\frac{1}{\xi}}), Q}^{\frac{1}{\xi}} \lesssim c_n 2^{n+2} \|b\|_{BMO}$$

where $\exp(L^{\frac{1}{\xi}})$ stands for the Young function $\psi(t) \approx \exp(t^{\frac{1}{\xi}}) - 1$.

Proof. By definition, we have

$$\begin{aligned} \| |b - b_Q|^\xi \|_{\exp(L^{\frac{1}{\xi}}), Q} &= \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \left[\exp\left(\frac{|b(x) - b_Q|}{\lambda^{\frac{1}{\xi}}}\right) - 1 \right] dx \leq 1 \right\} \\ &= \inf \left\{ \lambda^\xi > 0 : \frac{1}{|Q|} \int_Q \left[\exp\left(\frac{|b(x) - b_Q|}{\lambda}\right) - 1 \right] dx \leq 1 \right\} \\ &= \|b - b_Q\|_{\exp L, Q}^\xi \end{aligned}$$

which implies the desired estimate. \square

Through out this paper, we will make extensive use of the following proposition, which is actually a discrete Hölder inequality.

Proposition 3.9. *Suppose $p_1, p_2 > 1$, $p_3 > 0$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1 \leq \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$. We have the following inequality for non-negative sequences $\{a_j\}$, $\{b_j\}$, and $\{c_j\}$*

$$\sum_j a_j b_j c_j \leq \left(\sum_j a_j^{p_1} \right)^{\frac{1}{p_1}} \left(\sum_j b_j^{p_2} \right)^{\frac{1}{p_2}} \left(\sum_j c_j^{p_3} \right)^{\frac{1}{p_3}}.$$

4. PROOF OF THEOREM 2.3

Without loss of generality, we may assume that f and g are non-negative, bounded and compactly supported. By induction, we can prove that

$$\begin{aligned} & [\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x) \\ (4.1) \quad &= \int_{\mathbb{R}^n} \prod_{i=1}^m (b_i(x) - b_i(x-y)) \prod_{i=m+1}^N (b_i(x) - b_i(x+y)) \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy. \end{aligned}$$

For each $Q \in \mathcal{D}$, let $\lambda_i = \lambda_i(Q) = \int_{3Q} b_i(x) dx$ where $i = 1, \dots, N$, we have

$$\begin{aligned} \prod_{i=1}^m (b_i(x) - b_i(x-y)) &= \prod_{i=1}^m \left[(b_i(x) - \lambda_i) + (\lambda_i - b_i(x-y)) \right] \\ &= \sum_{A \subseteq \{1, \dots, m\}} \prod_{i \in A} (b_i(x) - \lambda_i) \prod_{i \in \bar{A}} (\lambda_i - b_i(x-y)) \end{aligned}$$

and similarly,

$$\prod_{i=m+1}^N (b_i(x) - b_i(x+y)) = \sum_{B \subseteq \{m+1, \dots, N\}} \prod_{i \in B} (b_i(x) - \lambda_i) \prod_{i \in \bar{B}} (\lambda_i - b_i(x+y)).$$

Hence

$$\begin{aligned} \prod_{i=1}^m (b_i(x) - b_i(x-y)) \prod_{i=m+1}^N (b_i(x) - b_i(x+y)) &= \\ \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \prod_{i \in A \cup B} (b_i(x) - \lambda_i) \prod_{i \in \bar{A}} (\lambda_i - b_i(x-y)) \prod_{i \in \bar{B}} (\lambda_i - b_i(x+y)). \end{aligned}$$

This estimate together with (4.1) yield

$$\begin{aligned} (4.2) \quad & |[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)| \\ & \leq \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \int_{\mathbb{R}^n} \prod_{i \in A \cup B} |b_i(x) - \lambda_i| \prod_{i \in \bar{A}} |b_i(x-y) - \lambda_i| \\ & \quad \prod_{i \in \bar{B}} |b_i(x+y) - \lambda_i| \frac{f(x-y)g(x+y)}{|y|^{n-\alpha}} dy \\ & \lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}-1} \int_{|y|_\infty \leq \ell(Q)} \prod_{i \in A \cup B} |b_i(x) - \lambda_i| \\ & \quad \prod_{i \in \bar{A}} |b_i(x-y) - \lambda_i| \prod_{i \in \bar{B}} |b_i(x+y) - \lambda_i| f(x-y)g(x+y) dy \chi_Q(x). \end{aligned}$$

Since $q \leq 1$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |[\vec{b}, \mathbf{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)|^q u(x) dx \\ & \lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{(\frac{\alpha}{n}-1)q} \\ & \quad \left[\int_Q \left[\int_{|y|_\infty \leq \ell(Q)} \prod_{i \in \bar{A}} |b_i(x-y) - \lambda_i| \prod_{i \in \bar{B}} |b_i(x+y) - \lambda_i| f(x-y)g(x+y) dy \right]^q \right. \\ & \quad \left. \left[\prod_{i \in A \cup B} |b_i(x) - \lambda_i| \right]^q u(x) dx. \right] \end{aligned}$$

If we use Hölder inequality with the pair $(\frac{1}{q}, \frac{1}{1-q})$, we will arrive at the inequality

$$\begin{aligned}
& \int_{\mathbb{R}^n} |[\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)|^q u(x) dx \\
& \lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{\left(\frac{\alpha}{n}-1\right)q} \\
& \quad \left[\int_Q \int_{|y|_\infty \leq \ell(Q)} \prod_{i \in \bar{A}} |b_i(x-y) - \lambda_i| \prod_{i \in \bar{B}} |b_i(x+y) - \lambda_i| f(x-y) g(x+y) dy dx \right]^q \\
& \quad \left[\int_Q \prod_{i \in A \cup B} |b_i(x) - \lambda_i|^{\frac{q}{1-q}} u(x)^{\frac{1}{1-q}} dx \right]^{1-q}.
\end{aligned}$$

By a change of variables, we have

$$\begin{aligned}
& (4.3) \\
& \int_{\mathbb{R}^n} |[\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)|^q u(x) dx \\
& \lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}q+1} \\
& \quad \left[\int_{3Q} \prod_{i \in \bar{A}} |b_i(t) - \lambda_i| f(t) dt \int_{3Q} \prod_{i \in \bar{B}} |b_i(z) - \lambda_i| g(z) dz \right]^q \\
& \quad \left[\int_{3Q} \prod_{i \in A \cup B} |b_i(x) - \lambda_i|^{\frac{q}{1-q}} u(x)^{\frac{1}{1-q}} dx \right]^{1-q}.
\end{aligned}$$

Now we use the generalized Hölder inequality, Theorem 3.7 and Corollary 3.8 to obtain the following estimates:

$$\begin{aligned}
\int_{3Q} \prod_{i \in \bar{A}} |b_i(t) - \lambda_i| f(t) dt & \lesssim \prod_{i \in \bar{A}} \|b_i - \lambda_i\|_{\exp L, 3Q} \|f\|_{L(\log L)^{|\bar{A}|}, 3Q} \\
& \lesssim \prod_{i \in \bar{A}} \|b_i\|_{BMO} \|f\|_{L(\log L)^{|\bar{A}|}, 3Q}
\end{aligned}$$

and

$$\begin{aligned}
\int_{3Q} \prod_{i \in \bar{B}} |b_i(z) - \lambda_i| g(z) dz & \lesssim \prod_{i \in \bar{B}} \|b_i - \lambda_i\|_{\exp L, 3Q} \|g\|_{L(\log L)^{|\bar{B}|}, 3Q} \\
& \lesssim \prod_{i \in \bar{B}} \|b_i\|_{BMO} \|g\|_{L(\log L)^{|\bar{B}|}, 3Q}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \int_{3Q} \prod_{i \in A \cup B} |b_i(x) - \lambda_i|^{\frac{q}{1-q}} u(x)^{\frac{1}{1-q}} dx \\
& \lesssim \prod_{i \in A \cup B} \left\| |b_i - \lambda_i|^{\frac{q}{1-q}} \right\|_{\exp(L^{\frac{1-q}{q}}), 3Q} \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{q|A \cup B|}{1-q}}, 3Q}
\end{aligned}$$

$$\lesssim \prod_{i \in A \cup B} \|b_i\|_{BMO}^{\frac{q}{1-q}} \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{q|A \cup B|}{1-q}}, 3Q}.$$

Substituting these estimates into (4.3) and use the facts: $|\bar{A}| \leq m$, $|\bar{B}| \leq N - m$, $|A \cup B| \leq N$, and stronger Young functions provide bigger Orlicz norms, we come up with the following estimates.

$$\begin{aligned} \int_{\mathbb{R}^n} |[\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)|^q u(x) dx &\lesssim \|\vec{b}\|^q \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}q+1} \\ &\quad \times \left(\|f\|_{L(\log L)^m, 3Q} \|g\|_{L(\log L)^{N-m}, 3Q} \right)^q \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, 3Q}^{1-q} \\ &\lesssim \|\vec{b}\|^q \sum_A \sum_B \sum_{t=1}^{3^n} \sum_{Q \in \mathcal{D}^t} |Q|^{\frac{\alpha}{n}q+1} \left(\|f\|_{L(\log L)^m, Q} \|g\|_{L(\log L)^{N-m}, Q} \right)^q \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q}^{1-q}. \end{aligned} \quad (4.4)$$

The last inequality in (4.4) comes from the fact that each $3Q$ is contained in a $Q_t \in \mathcal{D}^t$, $t \in \{1, 2, \dots, 3^n\}$, with the property $\ell(3Q) \leq \ell(Q_t) \leq 6\ell(3Q)$. We note here that each Q_t like that may contain more than one but at most 6^n such $3Q$'s where the Q 's are from a same layer of \mathcal{D} , and there are at most 3 possible layers. From here, it suffices to estimate inner most sum of the last expression in (4.4) for a generic dyadic grid \mathcal{D} . We will denote this sum as

$$S = \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}q+1} \left(\|f\|_{L(\log L)^m, Q} \|g\|_{L(\log L)^{N-m}, Q} \right)^q \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q}^{1-q}.$$

For simplicity, we will write \mathcal{D} to mean any of the \mathcal{D}^t 's, $t = 1, \dots, 3^n$.

Next, we will replace the sum over dyadic cubes with the sum over a spare family of Calderón-Zygmund cubes. More precisely, for each $k \in \mathbb{Z}$, let $\{Q_j^k\}_j$ be a collection of disjoint dyadic cubes that are maximal with respect to

$$\|f\|_{L(\log L)^m, Q_j^k} \|g\|_{L(\log L)^{N-m}, Q_j^k} > a^k,$$

where $a > 1$ will be chosen later. This is possible because $\|f\|_{L(\log L)^m, Q}$ and $\|g\|_{L(\log L)^{N-m}, Q}$ all tend to 0 as $\ell(Q)$ tends to ∞ . Let $\Omega_k = \bigcup_j Q_j^k$ and $E_j^k = Q_j^k \setminus \Omega_{k+1}$, so that the family $\{E_j^k\}_{j,k}$ is pairwise disjoint and $|Q_j^k| \leq 2|E_j^k|$. In fact,

$$\begin{aligned} |Q_j^k \cap \Omega_{k+1}| &= \sum_{Q_i^{k+1} \subseteq Q_j^k} |Q_i^{k+1}| \\ &\leq \frac{1}{a^{\frac{k+1}{2}}} \sum_i \left(|Q_i^{k+1}| \|f\|_{L(\log L)^m, Q_i^{k+1}} \right)^{\frac{1}{2}} \left(|Q_i^{k+1}| \|g\|_{L(\log L)^{N-m}, Q_i^{k+1}} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{a^{\frac{k+1}{2}}} \left(\sum_i |Q_i^{k+1}| \|f\|_{L(\log L)^m, Q_i^{k+1}} \right)^{\frac{1}{2}} \left(\sum_i |Q_i^{k+1}| \|g\|_{L(\log L)^{N-m}, Q_i^{k+1}} \right)^{\frac{1}{2}}. \end{aligned}$$

If $\lambda, \mu > 0$ then the previous sum is bounded by

$$\begin{aligned}
&\leq \frac{1}{a^{\frac{k+1}{2}}} \left[\sum_i |Q_i^{k+1}| \left(\lambda + \frac{\lambda}{|Q_i^{k+1}|} \int_{Q_i^{k+1}} \gamma_1 \left(\frac{|f|}{\lambda} \right) \right) \right]^{\frac{1}{2}} \\
&\quad \left[\sum_i |Q_i^{k+1}| \left(\mu + \frac{\mu}{|Q_i^{k+1}|} \int_{Q_i^{k+1}} \gamma_2 \left(\frac{|g|}{\mu} \right) \right) \right]^{\frac{1}{2}} \\
&= \frac{1}{a^{\frac{k+1}{2}}} \left[\sum_i \lambda \int_{Q_i^{k+1}} \left(1 + \gamma_1 \left(\frac{|f|}{\lambda} \right) \right) \right]^{\frac{1}{2}} \left[\sum_i \mu \int_{Q_i^{k+1}} \left(1 + \gamma_2 \left(\frac{|g|}{\mu} \right) \right) \right]^{\frac{1}{2}} \\
&\leq \frac{1}{a^{\frac{k+1}{2}}} \left[\lambda \int_{Q_j^k} \left(1 + \gamma_1 \left(\frac{|f|}{\lambda} \right) \right) \right]^{\frac{1}{2}} \left[\mu \int_{Q_j^k} \left(1 + \gamma_2 \left(\frac{|g|}{\mu} \right) \right) \right]^{\frac{1}{2}} \\
&\leq \frac{2^n}{a^{\frac{k+1}{2}}} |Q_j^k| \left[\lambda + \frac{\lambda}{|P|} \int_P \gamma_1 \left(\frac{|f|}{\lambda} \right) \right]^{\frac{1}{2}} \left[\mu + \frac{\mu}{|P|} \int_P \gamma_2 \left(\frac{|g|}{\mu} \right) \right]^{\frac{1}{2}}
\end{aligned}$$

where $\gamma_1(t) = t \log(e+t)^m$, $\gamma_2(t) = t \log(e+t)^{N-m}$, and P is an immediate dyadic parent of Q_j^k . By taking infimum over all $\lambda > 0$ and all $\mu > 0$, we have

$$\begin{aligned}
|Q_j^k \cap \Omega_{k+1}| &\leq \frac{2^{n+1}}{a^{\frac{k+1}{2}}} |Q_j^k| \left(\|f\|_{L(\log L)^m, P} \|g\|_{L(\log L)^{N-m}, P} \right)^{\frac{1}{2}} \\
&\leq \frac{2^{n+1}}{a^{\frac{k+1}{2}}} |Q_j^k| a^{\frac{k}{2}} \\
&= \frac{2^{n+1}}{a^{\frac{1}{2}}} |Q_j^k|
\end{aligned}$$

where the second inequality comes from the maximality of Q_j^k . With an appropriate choice of a , we will have $|Q_j^k \cap \Omega_{k+1}| \leq \frac{1}{2} |Q_j^k|$, and hence $|Q_j^k| \leq 2 |E_j^k|$ as we wish. Now, let

$$C_k = \{Q \in \mathcal{D} : a^k < \|f\|_{L(\log L)^m, Q} \|g\|_{L(\log L)^{N-m}, Q} \leq a^{k+1}\}$$

and notice that every $Q \in \mathcal{D}$ for which the summand of \mathbf{S} is non-zero must be in some C_k , and every $Q \in C_k$ is contained in a unique Q_j^k . So we have

$$\begin{aligned}
\mathbf{S} &\leq \sum_{k \in \mathbb{Z}} \sum_{Q \in C_k} |Q|^{\frac{\alpha}{n}q+1} \left(\|f\|_{L(\log L)^m, Q} \|g\|_{L(\log L)^{N-m}, Q} \right)^q \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q}^{1-q} \\
&\leq \sum_{k \in \mathbb{Z}} a^{(k+1)q} \sum_{Q \in C_k} |Q|^{\frac{\alpha}{n}q+1} \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q}^{1-q} \\
(4.5) \quad &\leq \sum_{k \in \mathbb{Z}} a^{(k+1)q} \sum_{j \in \mathbb{Z}} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_j^k}} |Q|^{\frac{\alpha}{n}q+1} \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q}^{1-q}.
\end{aligned}$$

For each $\lambda > 0$, the most inner sum is bounded by

$$\leq \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_j^k}} |Q|^{\frac{\alpha}{n}q+1} \left[\lambda + \frac{\lambda}{|Q|} \int_Q \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right]^{1-q},$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} |Q|^{\left(\frac{\alpha}{n}+1\right)q} \left[\lambda \int_Q \left(1 + \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right) \right]^{1-q} \\
&= |Q_j^k|^{\left(\frac{\alpha}{n}+1\right)q} \sum_{r=0}^{\infty} 2^{-qr\alpha - qrn} \sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} \left[\lambda \int_Q \left(1 + \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right) \right]^{1-q} \\
&\quad \times |Q_j^k|^{\left(\frac{\alpha}{n}+1\right)q} \sum_{r=0}^{\infty} 2^{-qr\alpha - qrn} \left[\sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} \lambda \int_Q \left(1 + \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right) \right]^{1-q} \\
&\quad \left(\sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} 1 \right)^q \\
&= |Q_j^k|^{\left(\frac{\alpha}{n}+1\right)q} \left[\lambda \int_{Q_j^k} \left(1 + \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right) \right]^{1-q} \sum_{r=0}^{\infty} 2^{-qr\alpha} \\
&= \frac{2^{\alpha q}}{2^{\alpha q} - 1} |Q_j^k|^{\frac{\alpha}{n}q+1} \left[\lambda + \frac{\lambda}{|Q_j^k|} \int_{Q_j^k} \gamma \left(\frac{|u^{\frac{1}{1-q}}|}{\lambda} \right) \right]^{1-q},
\end{aligned}$$

where $\gamma(t) = t \log(e+t)^{\frac{qN}{1-q}}$. By taking infimum over all $\lambda > 0$ and then substituting the result into (4.5), we have

$$\begin{aligned}
(4.6) \quad S &\lesssim \sum_{k \in \mathbb{Z}} a^{(k+1)q} \sum_{j \in \mathbb{Z}} |Q_j^k|^{\frac{\alpha}{n}q+1} \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q_j^k}^{1-q} \\
&\lesssim \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}q+1} \left(\|f\|_{L(\log L)^m, Q_j^k} \|g\|_{L(\log L)^{N-m}, Q_j^k} \right)^q \|u^{\frac{1}{1-q}}\|_{L(\log L)^{\frac{qN}{1-q}}, Q_j^k}^{1-q}.
\end{aligned}$$

Now we consider the following Young functions.

$$\tau_1(t) = \frac{t^{p_1}}{\log(e+t)^{1+(p_1-1)\delta}} \quad \text{and} \quad \tau_2(t) = \frac{t^{p_2}}{\log(e+t)^{1+(p_2-1)\delta}}.$$

Straightforward calculations show that $\tau_1 \in B_{p_1}$, $\tau_2 \in B_{p_2}$, and

$$\tau_1^{-1}(t) \phi_1^{-1}(t) \approx \frac{t}{\log(e+t)^m} \quad \text{and} \quad \tau_2^{-1}(t) \phi_2^{-1}(t) \approx \frac{t}{\log(e+t)^{N-m}}.$$

Using the generalized Hölder inequality and the imposed conditions on the weights, from (4.6) we have

$$\begin{aligned}
S &\lesssim \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}q+1} \left(\|f v_1^{\frac{1}{p_1}}\|_{\tau_1, Q_j^k} \|v_1^{-\frac{1}{p_1}}\|_{\phi_1, Q_j^k} \|g v_2^{\frac{1}{p_2}}\|_{\tau_2, Q_j^k} \|v_2^{-\frac{1}{p_2}}\|_{\phi_2, Q_j^k} \right)^q \\
&\quad \left\| u^{\frac{1}{1-q}} \right\|_{L(\log L)^{\frac{qN}{1-q}}, Q_j^k}^{1-q} \\
&\lesssim \sum_{k,j} |Q_j^k|^{\frac{q}{p}} \left(\|f v_1^{\frac{1}{p_1}}\|_{\tau_1, Q_j^k} \|g v_2^{\frac{1}{p_2}}\|_{\tau_2, Q_j^k} \right)^q.
\end{aligned}$$

From here, we are going to use: the condition that $p \leq q$, the fact that $|Q_j^k| \leq 2|E_j^k|$, discrete Hölder inequality with the pair $(\frac{p_1}{p}, \frac{p_2}{p})$, and theorem 3.2 to obtain the following estimates.

$$\begin{aligned}
(4.7) \quad S &\lesssim \left(\sum_{k,j} |Q_j^k| \|f v_1^{\frac{1}{p_1}}\|_{\tau_1, Q_j^k}^p \|g v_2^{\frac{1}{p_2}}\|_{\tau_2, Q_j^k}^p \right)^{\frac{q}{p}} \\
&\lesssim \left(\sum_{k,j} |E_j^k|^{\frac{p}{p_1}} \|f v_1^{\frac{1}{p_1}}\|_{\tau_1, Q_j^k}^p |E_j^k|^{\frac{p}{p_2}} \|g v_2^{\frac{1}{p_2}}\|_{\tau_2, Q_j^k}^p \right)^{\frac{q}{p}} \\
&\leq \left(\sum_{k,j} |E_j^k| \|f v_1^{\frac{1}{p_1}}\|_{\tau_1, Q_j^k}^{p_1} \right)^{\frac{q}{p_1}} \left(\sum_{k,j} |E_j^k| \|g v_2^{\frac{1}{p_2}}\|_{\tau_2, Q_j^k}^{p_2} \right)^{\frac{q}{p_2}} \\
&\leq \left(\sum_{k,j} \int_{E_j^k} M_{\tau_1} \left(f v_1^{\frac{1}{p_1}} \right) (x)^{p_1} dx \right)^{\frac{q}{p_1}} \left(\sum_{k,j} \int_{E_j^k} M_{\tau_2} \left(g v_2^{\frac{1}{p_2}} \right) (x)^{p_2} dx \right)^{\frac{q}{p_2}} \\
&\leq \left(\int_{\mathbb{R}^n} M_{\tau_1} \left(f v_1^{\frac{1}{p_1}} \right) (x)^{p_1} dx \right)^{\frac{q}{p_1}} \left(\int_{\mathbb{R}^n} M_{\tau_2} \left(g v_2^{\frac{1}{p_2}} \right) (x)^{p_2} dx \right)^{\frac{q}{p_2}} \\
&\lesssim \|f\|_{L^{p_1}(v_1)}^q \|g\|_{L^{p_2}(v_2)}^q.
\end{aligned}$$

Substituting the result in (4.7) into (4.4) will give us the desired estimate

$$\|[\vec{b}, \mathbf{B}\mathbf{I}_\alpha]_{\vec{\beta}}(f, g)\|_{L^q(u)} \lesssim \|\vec{b}\| \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}.$$

5. PROOF OF THEOREM 2.4

By duality, it suffices to prove that for all $f \in L^{p_1}(v_1)$, all $g \in L^{p_2}(v_2)$ and all $h \in L^{q'}(\mathbb{R}^n)$ with $\|h\|_{q'} = 1$,

$$\int_{\mathbb{R}^n} |[\vec{b}, \mathbf{B}\mathbf{I}_\alpha]_{\vec{\beta}}(f, g)(x)| h(x) u(x)^{\frac{1}{q}} dx \lesssim \|\vec{b}\| \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)}.$$

Without loss of generality, we may assume that f and g are non-negative, bounded and compactly supported. From (4.2), we have

$$\int_{\mathbb{R}^n} |[\vec{b}, \mathbf{B}\mathbf{I}_\alpha]_{\vec{\beta}}(f, g)(x)| h(x) u(x)^{\frac{1}{q}} dx$$

$$\begin{aligned}
&\lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{Q}} |Q|^{\frac{\alpha}{n}-1} \\
&\quad \int_Q \int_{|y| \leq \ell(Q)} \prod_{i \in \bar{A}} |b_i(x-y) - \lambda_i| \prod_{i \in \bar{B}} |b_i(x+y) - \lambda_i| f(x-y) g(x+y) dy \\
&\quad \prod_{i \in A \cup B} |b_i(x) - \lambda_i| h(x) u(x)^{\frac{1}{q}} dx.
\end{aligned}$$

If we use Hölder inequality with the pair (r, s) for the inner integral, and then perform a change in variables, we will get

$$\begin{aligned}
&\int_{\mathbb{R}^n} |[\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)| h(x) u(x)^{\frac{1}{q}} dx \\
&\lesssim \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{Q}} |Q|^{\frac{\alpha}{n}-1} \\
&\quad \left[\int_{3Q} \prod_{i \in \bar{A}} |b_i(t) - \lambda_i|^r f(t)^r dt \right]^{\frac{1}{r}} \left[\int_{3Q} \prod_{i \in \bar{B}} |b_i(z) - \lambda_i|^s g(z)^s dz \right]^{\frac{1}{s}} \\
&\quad \int_Q \prod_{i \in A \cup B} |b_i(x) - \lambda_i| h(x) u(x)^{\frac{1}{q}} dx.
\end{aligned} \tag{5.1}$$

We now use the generalized Hölder inequality, Theorem 3.7 and Corollary 3.8 to get the following estimates.

$$\begin{aligned}
&\int_{3Q} \prod_{i \in \bar{A}} |b_i(t) - \lambda_i|^r f(t)^r dt \lesssim \prod_{i \in \bar{A}} \| |b_i - \lambda_i|^r \|_{\exp(L^{\frac{1}{r}}), 3Q} \|f^r\|_{L(\log L)^{r|\bar{A}|}, 3Q} \\
&\lesssim \prod_{i \in \bar{A}} \|b_i\|_{BMO}^r \|f^r\|_{L(\log L)^{r|\bar{A}|}, 3Q} \\
&\int_{3Q} \prod_{i \in \bar{B}} |b_i(z) - \lambda_i|^s g(z)^s dz \lesssim \prod_{i \in \bar{B}} \| |b_i - \lambda_i|^s \|_{\exp(L^{\frac{1}{s}}), 3Q} \|g^s\|_{L(\log L)^{s|\bar{B}|}, 3Q} \\
&\lesssim \prod_{i \in \bar{B}} \|b_i\|_{BMO}^s \|g^s\|_{L(\log L)^{s|\bar{B}|}, 3Q} \\
&\int_Q \prod_{i \in A \cup B} |b_i(x) - \lambda_i| h(x) u(x)^{\frac{1}{q}} dx \lesssim \prod_{i \in A \cup B} \|b_i - \lambda_i\|_{\exp L, 3Q} \|hu^{\frac{1}{q}}\|_{L(\log L)^{|A \cup B|}, 3Q} \\
&\lesssim \prod_{i \in A \cup B} \|b_i\|_{BMO} \|hu^{\frac{1}{q}}\|_{L(\log L)^{|A \cup B|}, 3Q}.
\end{aligned}$$

Substituting these estimates into (5.1) and using the facts: $|\bar{A}| \leq m$, $|\bar{B}| \leq N-m$, $|A \cup B| \leq N$, and stronger Young functions provide bigger Orlicz norms, we come up with the following estimates.

$$\int_{\mathbb{R}^n} |[\vec{b}, \text{Bl}_\alpha]_{\vec{\beta}}(f, g)(x)| h(x) u(x)^{\frac{1}{q}} dx$$

$$\begin{aligned}
&\lesssim \|\vec{b}\| \sum_{A \subseteq \{1, \dots, m\}} \sum_{B \subseteq \{m+1, \dots, N\}} \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}+1} \|f^r\|_{L(\log L)^{mr}, 3Q}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, 3Q}^{\frac{1}{s}} \\
&\quad \times \|hu^{\frac{1}{q}}\|_{L(\log L)^N, 3Q} \\
&\lesssim \|\vec{b}\| \sum_A \sum_B \sum_{t=1}^{3^n} \sum_{Q \in \mathcal{D}} |Q|^{\frac{\alpha}{n}+1} \|f^r\|_{L(\log L)^{mr}, Q}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q}^{\frac{1}{s}} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q}.
\end{aligned}
\tag{5.2}$$

It suffices to control the inner most sum of the last expression in (5.2) for a general dyadic grid \mathcal{D} . To do so, we will replace the sum over dyadic cubes with the sum over a sparse family of Calderón-Zygmund cubes. Let $a > 1$ be a number that will be chosen later. For each $k \in \mathbb{Z}$, let $\{Q_j^k\}_j$ be a collection of disjoint dyadic cubes that are maximal with respect to

$$\|f^r\|_{L(\log L)^{mr}, Q}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q}^{\frac{1}{s}} > a^k.$$

Let $\Omega_k = \bigcup_j Q_j^k$ and $E_j^k = Q_j^k \setminus \Omega_{k+1}$, so that the family $\{E_j^k\}_{j,k}$ is pairwise disjoint and $|Q_j^k| \leq 2|E_j^k|$. In fact,

$$\begin{aligned}
|Q_j^k \cap \Omega_{k+1}| &= \sum_{Q_i^{k+1} \subseteq Q_j^k} |Q_i^{k+1}| \\
&\leq \frac{1}{a^{k+1}} \sum_i \left(|Q_i^{k+1}| \|f^r\|_{L(\log L)^{mr}, Q_i^{k+1}} \right)^{\frac{1}{r}} \left(|Q_i^{k+1}| \|g^s\|_{L(\log L)^{(N-m)s}, Q_i^{k+1}} \right)^{\frac{1}{s}} \\
&\leq \frac{1}{a^{k+1}} \left(\sum_i |Q_i^{k+1}| \|f^r\|_{L(\log L)^{mr}, Q_i^{k+1}} \right)^{\frac{1}{r}} \left(\sum_i |Q_i^{k+1}| \|g^s\|_{L(\log L)^{(N-m)s}, Q_i^{k+1}} \right)^{\frac{1}{s}}.
\end{aligned}$$

If $\lambda, \mu > 0$ then the previous sum is bounded by

$$\begin{aligned}
&\leq \frac{1}{a^{k+1}} \left[\sum_i |Q_i^{k+1}| \left(\lambda + \frac{\lambda}{|Q_i^{k+1}|} \int_{Q_i^{k+1}} \gamma_1 \left(\frac{|f|^r}{\lambda} \right) \right) \right]^{\frac{1}{r}} \\
&\quad \left[\sum_i |Q_i^{k+1}| \left(\mu + \frac{\mu}{|Q_i^{k+1}|} \int_{Q_i^{k+1}} \gamma_2 \left(\frac{|g|^s}{\mu} \right) \right) \right]^{\frac{1}{s}}, \quad \forall \mu > 0 \\
&= \frac{1}{a^{k+1}} \left[\sum_i \lambda \int_{Q_i^{k+1}} \left(1 + \gamma_1 \left(\frac{|f|^r}{\lambda} \right) \right) \right]^{\frac{1}{r}} \left[\sum_i \mu \int_{Q_i^{k+1}} \left(1 + \gamma_2 \left(\frac{|g|^s}{\mu} \right) \right) \right]^{\frac{1}{s}} \\
&\leq \frac{1}{a^{k+1}} \left[\lambda \int_{Q_j^k} \left(1 + \gamma_1 \left(\frac{|f|^r}{\lambda} \right) \right) \right]^{\frac{1}{r}} \left[\mu \int_{Q_j^k} \left(1 + \gamma_2 \left(\frac{|g|^s}{\mu} \right) \right) \right]^{\frac{1}{s}} \\
&\leq \frac{2^n}{a^{k+1}} |Q_j^k| \left[\lambda + \frac{\lambda}{|P|} \int_P \gamma_1 \left(\frac{|f|^r}{\lambda} \right) \right]^{\frac{1}{r}} \left[\mu + \frac{\mu}{|P|} \int_P \gamma_2 \left(\frac{|g|^s}{\mu} \right) \right]^{\frac{1}{s}}
\end{aligned}$$

where $\gamma_1(t) = t \log(e+t)^{mr}$, $\gamma_2(t) = t \log(e+t)^{(N-m)s}$, and P is an immediate dyadic parent of Q_j^k . By taking infimum over all $\lambda > 0$ and all $\mu > 0$, we have

$$\begin{aligned} |Q_j^k \cap \Omega_{k+1}| &\leq \frac{2^{n+1}}{a^{k+1}} |Q_j^k| \|f\|_{L(\log L)^{mr}, P}^{\frac{1}{r}} \|g\|_{L(\log L)^{(N-m)s}, P}^{\frac{1}{s}} \\ &\leq \frac{2^{n+1}}{a^{k+1}} |Q_j^k| a^k = \frac{2^{n+1}}{a} |Q_j^k| \end{aligned}$$

where the last inequality comes from the maximality of Q_j^k . With an appropriate choice of a , we will have $|Q_j^k| \leq 2|E_j^k|$. Now, let

$$C_k = \left\{ Q \in \mathcal{D} : a^k < \|f^r\|_{L(\log L)^{mr}, Q}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q}^{\frac{1}{s}} \leq a^{k+1} \right\}$$

and notice that every $Q \in \mathcal{D}$ for which the summand of S is non-zero must be in some C_k , and every $Q \in C_k$ is contained in a unique Q_j^k . So we have

$$\begin{aligned} S &\leq \sum_{k \in \mathbb{Z}} \sum_{Q \in C_k} |Q|^{\frac{\alpha}{n}+1} \|f^r\|_{L(\log L)^{mr}, Q}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q}^{\frac{1}{s}} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q} \\ (5.3) \quad &\leq \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{Q \in C_k} |Q|^{\frac{\alpha}{n}+1} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q} \\ &\leq \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{j \in \mathbb{Z}} \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_j^k}} |Q|^{\frac{\alpha}{n}+1} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q}. \end{aligned}$$

For all $\lambda > 0$ the most inner sum is bounded by

$$\begin{aligned} &\leq \sum_{\substack{Q \in \mathcal{D} \\ Q \subseteq Q_j^k}} |Q|^{\frac{\alpha}{n}+1} \left[\lambda + \frac{\lambda}{|Q|} \int_Q \gamma \left(\frac{|hu^{\frac{1}{q}}|}{\lambda} \right) \right], \\ &= \sum_{r=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} |Q|^{\frac{\alpha}{n}} \lambda \int_Q \left[1 + \gamma \left(\frac{|hu^{\frac{1}{q}}|}{\lambda} \right) \right] \\ &= \lambda |Q_j^k|^{\frac{\alpha}{n}} \sum_{r=0}^{\infty} 2^{-r\alpha} \sum_{\substack{Q \in \mathcal{D}, Q \subseteq Q_j^k \\ \ell(Q)=2^{-r}\ell(Q_j^k)}} \int_Q \left[1 + \gamma \left(\frac{|hu^{\frac{1}{q}}|}{\lambda} \right) \right] \\ &= \frac{2^\alpha}{2^\alpha - 1} |Q_j^k|^{\frac{\alpha}{n}+1} \left[\lambda + \frac{\lambda}{|Q_j^k|} \int_{Q_j^k} \gamma \left(\frac{|hu^{\frac{1}{q}}|}{\lambda} \right) \right], \end{aligned}$$

where $\gamma(t) = t \log(e+t)^N$. By taking infimum over all $\lambda > 0$ and then substituting the result into (5.3), we end up having

$$\begin{aligned} S &\lesssim \sum_{k \in \mathbb{Z}} a^{k+1} \sum_{j \in \mathbb{Z}} |Q_j^k|^{\frac{\alpha}{n}+1} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q_j^k} \\ (5.4) \quad &\lesssim \sum_{k, j} |Q_j^k|^{\frac{\alpha}{n}+1} \|f^r\|_{L(\log L)^{mr}, Q_j^k}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q_j^k}^{\frac{1}{s}} \|hu^{\frac{1}{q}}\|_{L(\log L)^N, Q_j^k}. \end{aligned}$$

Now we consider the following Young functions.

$$\begin{aligned}\tau_1(t) &= \frac{t^{\frac{p_1}{r}}}{\log(e+t)^{1+(\frac{p_1}{r}-1)\delta}} \\ \tau_2(t) &= \frac{t^{\frac{p_2}{s}}}{\log(e+t)^{1+(\frac{p_2}{s}-1)\delta}} \\ \tau(t) &= \frac{t^{q'}}{\log(e+t)^{1+(q'-1)\delta}}\end{aligned}$$

Straight forward calculations show that $\tau_1 \in B_{\frac{p_1}{r}}$, $\tau_2 \in B_{\frac{p_2}{s}}$, $\tau \in B_{q'}$, and

$$\begin{aligned}\tau_1^{-1}(t) \phi_1^{-1}(t) &\approx \frac{t}{\log(e+t)^{mr}} \\ \tau_2^{-1}(t) \phi_2^{-1}(t) &\approx \frac{t}{\log(e+t)^{(N-m)s}} \\ \tau^{-1}(t) \psi^{-1}(t) &\approx \frac{t}{\log(e+t)^N}.\end{aligned}$$

So, by using the generalized Hölder inequality and the imposed conditions on the weights, from (5.4) we have

$$\begin{aligned}S &\lesssim \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}+1} \|f^r v_1^{\frac{r}{p_1}}\|_{\tau_1, Q_j^k}^{\frac{1}{r}} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q_j^k}^{\frac{1}{r}} \|g^s v_2^{\frac{s}{p_2}}\|_{\tau_2, Q_j^k}^{\frac{1}{s}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q_j^k}^{\frac{1}{s}} \\ &\quad \|h\|_{\tau, Q_j^k} \|u^{\frac{1}{q}}\|_{\psi, Q_j^k} \\ &\lesssim \sum_{k,j} |Q_j^k|^{\frac{1}{p}+\frac{1}{q'}} \|f^r v_1^{\frac{r}{p_1}}\|_{\tau_1, Q_j^k}^{\frac{1}{r}} \|g^s v_2^{\frac{s}{p_2}}\|_{\tau_2, Q_j^k}^{\frac{1}{s}} \|h\|_{\tau, Q_j^k}.\end{aligned}$$

We are going to use: the fact that $|Q_j^k| \leq 2|E_j^k|$, Proposition 3.9 with the triple (p_1, p_2, q') , and Theorem 3.2 to obtain the following estimates.

$$\begin{aligned}S &\lesssim \sum_{k,j} \left(\|f^r v_1^{\frac{r}{p_1}}\|_{\tau_1, Q_j^k}^{\frac{1}{r}} |E_j^k|^{\frac{1}{p_1}} \right) \left(\|g^s v_2^{\frac{s}{p_2}}\|_{\tau_2, Q_j^k}^{\frac{1}{s}} |E_j^k|^{\frac{1}{p_2}} \right) \left(\|h\|_{\tau, Q_j^k} |E_j^k|^{\frac{1}{q'}} \right) \\ &\leq \left[\sum_{k,j} \|f^r v_1^{\frac{r}{p_1}}\|_{\tau_1, Q_j^k}^{\frac{p_1}{r}} |E_j^k| \right]^{\frac{1}{p_1}} \left[\sum_{k,j} \|g^s v_2^{\frac{s}{p_2}}\|_{\tau_2, Q_j^k}^{\frac{p_2}{s}} |E_j^k| \right]^{\frac{1}{p_2}} \left[\sum_{k,j} \|h\|_{\tau, Q_j^k}^{q'} |E_j^k| \right]^{\frac{1}{q'}}\end{aligned}\tag{5.5}$$

$$\begin{aligned}
& \leq \left[\sum_{k,j} \int_{E_j^k} M_{\tau_1} \left(f^r v_1^{\frac{r}{p_1}} \right) (x)^{\frac{p_1}{r}} dx \right]^{\frac{1}{p_1}} \left[\sum_{k,j} \int_{E_j^k} M_{\tau_2} \left(g^s v_2^{\frac{s}{p_2}} \right) (x)^{\frac{p_2}{s}} dx \right]^{\frac{1}{p_2}} \\
& \quad \left[\sum_{k,j} \int_{E_j^k} M_{\tau} (h) (x)^{q'} dx \right]^{\frac{1}{q'}} \\
& \leq \left[\int_{\mathbb{R}^n} M_{\tau_1} \left(f^r v_1^{\frac{r}{p_1}} \right) (x)^{\frac{p_1}{r}} dx \right]^{\frac{1}{p_1}} \left[\int_{\mathbb{R}^n} M_{\tau_2} \left(g^s v_2^{\frac{s}{p_2}} \right) (x)^{\frac{p_2}{s}} dx \right]^{\frac{1}{p_2}} \\
& \quad \left[\int_{\mathbb{R}^n} M_{\tau} (h) (x)^{q'} dx \right]^{\frac{1}{q'}} \\
& \lesssim \|f\|_{L^{p_1}(v_1)} \|g\|_{L^{p_2}(v_2)} \|h\|_{q'}.
\end{aligned}$$

Since S can be any term in (5.2), and the number of terms in (5.2) is finite, substituting the result in (5.5) into (5.2) will complete our proof.

6. PROOF OF THEOREM 2.6

Again, without loss of generality, we may restrict ourselves onto working with f and g that are non-negative, bounded and compactly supported. Thanks to Theorem 3.6, we only need to verify the inequality for a certain $q_0 \in (0, \infty)$ and an arbitrary weight $w \in A_\infty$. We will work with $q_0 = 1$. By mimicking what we did in the proof of Theorem 2.4, we have

$$\begin{aligned}
(6.1) \quad & \int_{\mathbb{R}^n} |[\vec{b}, \mathbf{BT}_\alpha]_{\vec{\beta}}(f, g)(x)| w(x) dx \\
& \lesssim \|\vec{b}\| \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}+1} \|f^r\|_{L(\log L)^{mr}, Q_j^k}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q_j^k}^{\frac{1}{s}} \|w\|_{L(\log L)^N, Q_j^k}.
\end{aligned}$$

Since $w \in A_\infty$, there exist, by Lemma 3.4, a number $m > 1$ such that

$$\left(\int_Q w^m \right)^{\frac{1}{m}} \lesssim \int_Q w.$$

The Young function $\psi(t) = t^m$ is stronger than $\phi(t) = t \log(e+t)^N$, which implies

$$\|w\|_{L(\log L)^N, Q_j^k} \lesssim \left(\int_Q w^m \right)^{\frac{1}{m}} \lesssim \int_Q w.$$

Substituting this result into (6.1), we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |[\vec{b}, \mathbf{B}\mathbf{T}_\alpha]_{\vec{\beta}}(f, g)(x)| w(x) dx \\
& \lesssim \|\vec{b}\| \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}} \|f^r\|_{L(\log L)^{mr}, Q_j^k}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q_j^k}^{\frac{1}{s}} w(Q_j^k) \\
& \lesssim \|\vec{b}\| \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}} \|f^r\|_{L(\log L)^{mr}, Q_j^k}^{\frac{1}{r}} \|g^s\|_{L(\log L)^{(N-m)s}, Q_j^k}^{\frac{1}{s}} w(E_j^k) \\
& \leq \|\vec{b}\| \sum_{k,j} \int_{E_j^k} \mathcal{M}_\alpha^{r,s}(f, g)(x) w(x) dx \\
& \leq \|\vec{b}\| \int_{\mathbb{R}^n} \mathcal{M}_\alpha^{r,s}(f, g)(x) w(x) dx
\end{aligned}$$

where the second inequality is due to Lemma 3.4 and the fact that $w \in A_\infty$.

7. PROOF OF THEOREM 2.7

[Condition (2.1) \Rightarrow the weak type boundedness]

In light of Theorem 3.1, it is not hard to see

$$\mathcal{M}_\alpha^{r,s}(f, g)(x) \leq 6^{n-\alpha} \sum_{t \in \{0, 1/3\}^n} \mathcal{M}_\alpha^{r,s, \mathcal{D}^t}(f, g)(x)$$

where

$$\mathcal{M}_\alpha^{r,s, \mathcal{D}}(f, g)(x) = \sup_{\mathcal{D} \ni Q \ni x} |Q|^{\frac{\alpha}{n}} \left(\int_Q |f|^r \right)^{\frac{1}{r}} \left(\int_Q |g|^s \right)^{\frac{1}{s}}.$$

So, we will only need to prove the weak type boundedness for $\mathcal{M}_\alpha^{r,s, \mathcal{D}}$ where \mathcal{D} is an arbitrary dyadic grid. Without loss of generality, we may assume that f, g are non-negative, bounded and compactly supported. By performing the Calderón-Zygmund decomposition algorithm, we have

$$(7.1) \quad E_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}_\alpha^{r,s, \mathcal{D}}(f, g)(x) > \lambda\} = \bigcup_j Q_j$$

where Q_j 's are pairwise disjoint maximal dyadic cubes that satisfy

$$(7.2) \quad |Q_j|^{\frac{\alpha}{n}} \left(\int_{Q_j} f^r \right)^{\frac{1}{r}} \left(\int_{Q_j} g^s \right)^{\frac{1}{s}} > \lambda.$$

From (7.1) and (7.2) we have

$$\begin{aligned}
u(E_\lambda) &= \sum_j \int_{Q_j} u = \sum_j |Q_j| \int_{Q_j} u \\
&\leq \frac{1}{\lambda^q} \sum_j |Q_j|^{\frac{q\alpha}{n}+1} \left(\int_{Q_j} u \right) \left(\int_{Q_j} f^r \right)^{\frac{q}{r}} \left(\int_{Q_j} g^s \right)^{\frac{q}{s}} \\
&\leq \frac{1}{\lambda^q} \left[\sum_j |Q_j|^{\frac{p\alpha}{n}+\frac{p}{q}} \left(\int_{Q_j} u \right)^{\frac{p}{q}} \left(\int_{Q_j} f^r \right)^{\frac{p}{r}} \left(\int_{Q_j} g^s \right)^{\frac{p}{s}} \right]^{\frac{q}{p}}.
\end{aligned}$$

By using Hölder inequality for the second and the third dashed integrals and then using condition (2.1), we obtain

$$\begin{aligned}
u(E_\lambda) &\leq \frac{1}{\lambda^q} \left[\sum_j |Q_j|^{\frac{p\alpha}{n} + \frac{p}{q} - 1} \left(\int_{Q_j} u \right)^{\frac{p}{q}} \left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{p_1 r} p} \left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{p_2 s} p} \right. \\
&\quad \left. \left(\int_{Q_j} f^{p_1} v_1 \right)^{\frac{p}{p_1}} \left(\int_{Q_j} g^{p_2} v_2 \right)^{\frac{p}{p_2}} \right]^{\frac{q}{p}} \\
&\lesssim \frac{1}{\lambda^q} \left[\sum_j \left(\int_{Q_j} f^{p_1} v_1 \right)^{\frac{p}{p_1}} \left(\int_{Q_j} g^{p_2} v_2 \right)^{\frac{p}{p_2}} \right]^{\frac{q}{p}} \\
&\leq \frac{1}{\lambda^q} \left(\sum_j \int_{Q_j} f^{p_1} v_1 \right)^{\frac{q}{p_1}} \left(\sum_j \int_{Q_j} g^{p_2} v_2 \right)^{\frac{q}{p_2}} \\
&\leq \frac{1}{\lambda^q} \|f\|_{L^{p_1}(v_1)}^q \|g\|_{L^{p_2}(v_2)}^q.
\end{aligned}$$

[The weak type boundedness \Rightarrow condition (2.1)]

For any cube Q , let $f = v_1^{-\frac{1}{p_1-r}} \chi_Q$ and $g = v_2^{-\frac{1}{p_2-s}} \chi_Q$.

If $\left(\int_Q f^r \right)^{\frac{1}{r}} \left(\int_Q g^s \right)^{\frac{1}{s}} > 0$, then by choosing $\lambda = \frac{1}{2} |Q|^{\frac{\alpha}{n}} \left(\int_Q f^r \right)^{\frac{1}{r}} \left(\int_Q g^s \right)^{\frac{1}{s}}$, from the weak type boundedness we have

$$u(Q)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n}} \left(\int_Q f^r \right)^{\frac{1}{r}} \left(\int_Q g^s \right)^{\frac{1}{s}} \leq 2C \left(\int_{\mathbb{R}^n} f^{p_1} v_1 \right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{\frac{1}{p_2}}.$$

Now, if we substitute our specific choices of f and g into the expression, we have

$$u(Q)^{\frac{1}{q}} |Q|^{\frac{\alpha}{n} - 1} \left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{1}{r} - \frac{1}{p_1}} \left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{1}{s} - \frac{1}{p_2}} \leq 2C$$

which is equivalent to

$$|Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q u \right)^{\frac{1}{q}} \left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{p_1 r}} \left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{p_2 s}} \leq 2C$$

and this finishes the proof.

8. PROOF OF THEOREM 2.8

As explained previously, we only need to treat the dyadic operator $\mathcal{M}_\alpha^{r,s,\mathcal{D}}$, and work with non-negative, bounded and compactly supported functions f and g . Let $a > 1$ to be chosen later. For each $k \in \mathbb{Z}$, we have

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}_\alpha^{r,s,\mathcal{D}}(f,g)(x) > a^k\} = \bigcup_j Q_j^k$$

where Q_j^k 's are pairwise disjoint maximal dyadic cubes satisfying

$$|Q_j^k|^{\frac{\alpha}{n}} \left(\int_{Q_j^k} f^r \right)^{\frac{1}{r}} \left(\int_{Q_j^k} g^s \right)^{\frac{1}{s}} > a^k.$$

Let $E_j^k = Q_j^k \setminus \Omega_{k+1}$, then by a similar (in fact easier) argument as in the proof of theorem 2.4 we have a disjoint family $\{E_j^k\}_{k,j}$ and that $|Q_j^k| \leq 2|E_j^k|$ with an appropriate choice of a . For any $h \in L^{q'}(\mathbb{R}^n)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \mathcal{M}_\alpha^{r,s,\mathcal{D}}(f,g)(x) h(x) u(x)^{\frac{1}{q}} dx \\ &= \sum_{k \in \mathbb{Z}} \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}_\alpha^{r,s,\mathcal{D}}(f,g)(x) h(x) u(x)^{\frac{1}{q}} dx \\ &\leq \sum_{k \in \mathbb{Z}} a^{k+1} \sum_j \int_{Q_j^k} h(x) u(x)^{\frac{1}{q}} dx \\ &\leq a \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}} \left(\int_{Q_j^k} f^r \right)^{\frac{1}{r}} \left(\int_{Q_j^k} g^s \right)^{\frac{1}{s}} \int_{Q_j^k} h(x) u(x)^{\frac{1}{q}} dx \\ &\lesssim \sum_{k,j} |Q_j^k|^{\frac{\alpha}{n}+1} \|f^r v_1^{\frac{r}{p_1}}\|_{\bar{\phi}_1, Q_j^k}^{\frac{1}{r}} \|v_1^{\frac{-r}{p_1}}\|_{\bar{\phi}_1, Q_j^k}^{\frac{1}{r}} \|g^s v_2^{\frac{s}{p_2}}\|_{\bar{\phi}_2, Q_j^k}^{\frac{1}{s}} \|v_2^{\frac{-s}{p_2}}\|_{\bar{\phi}_2, Q_j^k}^{\frac{1}{s}} \\ &\quad \|h\|_{\bar{\psi}, Q_j^k} \|u^{\frac{1}{q}}\|_{\psi, Q_j^k} \\ &\lesssim \sum_{k,j} \left(\|f^r v_1^{\frac{r}{p_1}}\|_{\bar{\phi}_1, Q_j^k}^{\frac{1}{r}} |E_j^k|^{\frac{1}{p_1}} \right) \left(\|g^s v_2^{\frac{s}{p_2}}\|_{\bar{\phi}_2, Q_j^k}^{\frac{1}{s}} |E_j^k|^{\frac{1}{p_2}} \right) \\ &\quad \left(\|h\|_{\bar{\psi}, Q_j^k} |E_j^k|^{\frac{1}{q'}} \right). \end{aligned}$$

From here, our argument will just be similar to that in (5.5), where we will need to use the assumptions: $\bar{\psi} \in B_{q'}$, $\bar{\phi}_1 \in B_{\frac{p_1}{r}}$ and $\bar{\phi}_2 \in B_{\frac{p_2}{s}}$.

9. PROOF OF THEOREM 2.9

When $u^{\frac{1}{q}} = v_1^{\frac{1}{p_1}} v_2^{\frac{1}{p_2}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, condition (2.1) becomes

$$(9.1) \quad \sup_Q \left(\int_Q v_1^{\frac{q}{p_1}} v_2^{\frac{q}{p_2}} \right)^{\frac{1}{q}} \left(\int_Q v_1^{-\frac{r}{p_1-r}} \right)^{\frac{p_1-r}{p_1 r}} \left(\int_Q v_2^{-\frac{s}{p_2-s}} \right)^{\frac{p_2-s}{p_2 s}} < \infty$$

which implies

$$u \in A_{2q} \quad \text{and} \quad v_1^{-\frac{r}{p_1-r}} \in A_{\frac{2p_1 r}{p_1-r}} \quad \text{and} \quad v_2^{-\frac{s}{p_2-s}} \in A_{\frac{2p_2 s}{p_2-s}}$$

by using theorem 3.5. Then by theorem 3.4, there exists $m > 1$ such that

$$\begin{aligned}
\left(\int_Q u^m\right)^{\frac{1}{mq}} &\leq \left(\int_Q u\right)^{\frac{1}{q}} \\
\left(\int_Q v_1^{-\frac{mr}{p_1-r}}\right)^{\frac{p_1-r}{mp_1r}} &\leq \left(\int_Q v_1^{-\frac{r}{p_1-r}}\right)^{\frac{p_1-r}{p_1r}} \\
\left(\int_Q v_2^{-\frac{ms}{p_2-s}}\right)^{\frac{p_2-s}{mp_2s}} &\leq \left(\int_Q v_2^{-\frac{s}{p_2-s}}\right)^{\frac{p_2-s}{p_2s}}.
\end{aligned}$$

These inequalities together with (9.1) imply

$$(9.2) \quad \sup_Q \left(\int_Q u^m\right)^{\frac{1}{mq}} \left(\int_Q v_1^{-\frac{mr}{p_1-r}}\right)^{\frac{p_1-r}{mp_1r}} \left(\int_Q v_2^{-\frac{ms}{p_2-s}}\right)^{\frac{p_2-s}{mp_2s}} < \infty.$$

Now, if we consider the Young functions: $\psi(t) = t^{mq}$, $\phi_1(t) = t^{\frac{mp_1}{p_1-r}}$ and $\phi_2(t) = t^{\frac{mp_2}{p_2-s}}$, then we have $\bar{\psi} \in B_{q'}$, $\bar{\phi}_1 \in B_{\frac{p_1}{r}}$ and $\bar{\phi}_2 \in B_{\frac{p_2}{s}}$. Moreover, we can reformulate (9.2) as

$$\sup_Q |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \|u^{\frac{1}{q}}\|_{\psi, Q} \|v_1^{-\frac{r}{p_1}}\|_{\phi_1, Q}^{\frac{1}{r}} \|v_2^{-\frac{s}{p_2}}\|_{\phi_2, Q}^{\frac{1}{s}} < \infty$$

where we used the Sobolev condition $\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p} = 0$. This is exactly the condition on the weights (u, v_1, v_2) in Theorem 2.2, so the conclusion is immediate.

10. APPLICATIONS AND EXAMPLES

In [19] Stein and Weiss proved the following inequality:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\gamma |x-y|^\alpha |y|^\beta} dx dy \lesssim \|f\|_p \|g\|_{q'}$$

where α, β , and γ are positive numbers that depend on p and q . Below we have a bilinear Stein-Weiss inequality for the case when $1 < p \leq q < \infty$. The case when $\frac{1}{2} < p \leq q \leq 1$ was done by the second author [14].

Theorem 10.1. *Suppose $1 < p_1, p_2 < \infty$ and $1 < p \leq q < \infty$. If $\alpha, \beta, \gamma_1, \gamma_2$ satisfy*

$$(10.1) \quad \beta < \frac{n}{q}, \quad \gamma_1 < (p-1)\frac{n}{p_1}, \quad \gamma_2 < (p-1)\frac{n}{p_2}$$

$$(10.2) \quad \alpha + \beta + \gamma_1 + \gamma_2 = n + \frac{n}{q} - \frac{n}{p}$$

$$(10.3) \quad \beta + \gamma_1 + \gamma_2 \geq 0$$

Then we have

$$(10.4) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)h(x)}{|y|^\alpha |x-y|^{\gamma_1} |x+y|^{\gamma_2} |x|^\beta} dx dy \lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{q'}$$

for non-negative functions f, g , and h .

Remark 10.2. Condition (10.1) corresponds to the condition

$$\beta < (1-q)\frac{n}{q}, \quad \gamma_1 < \frac{n}{p_1}, \quad \gamma_2 < \frac{n}{p_2}$$

which is stated in [14]. The interesting phenomenon here is that the factor $1-q$ (for the case $p \leq q \leq 1$) has become $p-1$ (for the case $1 < p \leq q$). This may reveal some clues about the case $p \leq 1 < q$.

Remark 10.3. If we think of the linear case just as a restriction of the bilinear one, then we can just drop p_2 and γ_2 , and identify p_1 with p , γ_1 with γ . At that time, conditions (10.1)-(10.3) will become

$$\begin{aligned} \beta &< \frac{n}{q}, \quad \gamma_1 < \frac{n}{p'} \\ \alpha + \beta + \gamma &= n + \frac{n}{q} - \frac{n}{p} \\ \beta + \gamma &\geq 0 \end{aligned}$$

which are exactly the needed conditions for the (linear) Stein-Weiss inequality to hold true.

Remark 10.4. The inequality (10.4) can also be written as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) g(y) h\left(\frac{x+y}{2}\right)}{|x-y|^\alpha |x|^{\gamma_1} |y|^{\gamma_2} |x+y|^\beta} dx dy \lesssim \|f\|_{p_1} \|g\|_{p_2} \|h\|_{q'}.$$

Proof of Theorem 10.1. Inequality (10.4) is just a dualized form of the following inequality with some appropriate weight-scalings on the functions f and g .

$$\begin{aligned} &\left[\int_{\mathbb{R}^n} \left(|\text{Bl}_{n-\alpha}(f, g)(x)| |x|^{-\beta} \right)^q dx \right]^{\frac{1}{q}} \\ &\lesssim \left[\int_{\mathbb{R}^n} \left(|f(x)| |x|^{\gamma_1} \right)^{p_1} dx \right]^{\frac{1}{p_1}} \left[\int_{\mathbb{R}^n} \left(|g(x)| |x|^{\gamma_2} \right)^{p_2} dx \right]^{\frac{1}{p_2}}. \end{aligned}$$

We are going to apply theorem 2.2 here, so we only need to check condition (2.1) with $u = |x|^{-q\beta}$, $v_1 = |x|^{p_1\gamma_1}$, $v_2 = |x|^{p_2\gamma_2}$, $r = \frac{p_1}{p}$ and $s = \frac{p_2}{p}$. To be clearer, we need to show that

$$(10.5) \quad \sup_Q |Q|^{\frac{n-\alpha}{n} + \frac{1}{q} - \frac{1}{p}} \left(\int_Q |x|^{-\beta q} \right)^{\frac{1}{q}} \left(\int_Q |x|^{\frac{\gamma_1 p_1}{p-1}} \right)^{\frac{p-1}{p_1}} \left(\int_Q |x|^{\frac{\gamma_2 p_2}{p-1}} \right)^{\frac{p-1}{p_2}} < \infty.$$

Now, for any cube Q , let $Q_0 = Q(O, \ell(Q))$. We then either have $Q \cap Q_0 = \emptyset$ or $Q \cap Q_0 \neq \emptyset$. If $Q \cap Q_0 = \emptyset$, then $|x| \sim |x|_\infty \geq \ell(Q)$ for all $x \in Q$. This implies that the left hand side of (10.5) is bounded by

$$\sup_Q |\ell(Q)|^{n-\alpha + \frac{n}{q} - \frac{n}{p}} |\ell(Q)|^{-\beta - \gamma_1 - \gamma_2} = 1.$$

If $Q \cap Q_0 \neq \emptyset$, then $|x| \leq \sqrt{n} |x|_\infty \leq 2\sqrt{n} \ell(Q)$ for all $x \in Q$. This implies that $Q \subset B = B(O, 2\sqrt{n} \ell(Q))$, and hence the left hand side of (10.5) is bounded by a constant. \square

Finally we end the paper with an example to show that our condition, condition (2.5), on the weights for BM is more general than the known results $(w_1, w_2) \in A_p \times A_p$. In fact we will show that there exists weights (w_1, w_2) that satisfy (2.5) but $w_1 \notin A_p$ and $w_2 \notin A_p$. Here, we are going to give an example of weights w_1 , and w_2 for this fact. Consider $w_1 = |x|^\alpha$, and $w_2 = |x|^\beta$. We shall prove

$$K = \sup_Q \left(\int_Q |x|^{\frac{p\alpha}{p_1} + \frac{p\beta}{p_2}} \right)^{\frac{1}{p}} \left(\int_Q |x|^{\frac{\alpha}{1-p}} \right)^{\frac{p-1}{p_1}} \left(\int_Q |x|^{\frac{\beta}{1-p}} \right)^{\frac{p-1}{p_2}} < \infty.$$

For every cube Q , we have 2 situations: either $|c_Q|_\infty \leq 2\ell(Q)$ or $|c_Q|_\infty > 2\ell(Q)$. If $|c_Q|_\infty \leq 2\ell(Q)$, then

$$\begin{aligned} K &\leq \ell(Q)^{-n} \left(\int_{B_0} |x|^{\frac{p\alpha}{p_1} + \frac{p\beta}{p_2}} \right)^{\frac{1}{p}} \left(\int_{B_0} |x|^{\frac{\alpha}{1-p}} \right)^{\frac{p-1}{p_1}} \left(\int_{B_0} |x|^{\frac{\beta}{1-p}} \right)^{\frac{p-1}{p_2}} \\ &\approx \ell(Q)^{-n + \frac{\alpha}{p_1} + \frac{\beta}{p_2} + \frac{n}{p} + \frac{n(p-1)-\alpha}{p_1} + \frac{n(p-1)-\beta}{p_2}} = 1 \end{aligned}$$

where $B_0 = B(3\sqrt{n}\ell(Q))$, and whenever $\alpha < n(p-1)$, $\beta < n(p-1)$, $-n < \frac{p\alpha}{p_1} + \frac{p\beta}{p_2}$.

If $|c_Q|_\infty > 2\ell(Q)$, then $|x| \sim |x|_\infty \sim |c_Q|_\infty \sim |c_Q|$ and hence

$$K \approx |c_Q|^{\frac{\alpha}{p_1} + \frac{\beta}{p_2} - \frac{\alpha}{p_1} - \frac{\beta}{p_2}} = 1.$$

These mean that $K < \infty$ whenever $\alpha < n(p-1)$, $\beta < n(p-1)$, $-n < \frac{p\alpha}{p_1} + \frac{p\beta}{p_2}$. So, we may have α get close to $-n(1+p_1-p)$, which is less than $-n$, as long as $\beta < n(p-1)$. Similarly, we may have β get close to $-n(1+p_2-p)$, which is less than $-n$, as long as $\alpha < n(p-1)$. This fact provides a wider range for α and β because the $A_p \times A_p$ requires $-n < \alpha, \beta < n(1-p)$.

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